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Stratified laminar flows in a circular pipe: New analytical solutions in terms of elementary functions

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ABSTRACT

Laminar stratified two-phase flows in circular pipes are studied. Under the assumption that the wall-liquid wettability properties of the phases are identical, new exact analytical solutions in terms of elementary functions are constructed. The solutions satisfy the Navier–Stokes equations exactly in fluids and the boundary conditions on the pipe walls and the interface for the two cases: the pipe is horizontal and the capillarity forces dominate the gravity ones, the pipe is inclined and the volumetric quantities of liquids in the pipe are the same. For the second case, the capillarity and gravity forces can be arbitrary, but if the gravity forces dominate the capillarity ones, the assumption about the equal wall-liquid tensions of the phases can be withdrawn.

1. Introduction

Stratified laminar flows in circular pipes have been studied for several decades. The interest in this subject is caused by two circumstances. First, there exists a possibility of obtaining exact analytical solutions to the Navier–Stokes equations, which is always of theoretical interest, and second, there are a number of practical applications connected mainly with the lubrication effect of water in pumping oil through circular conduits.

It seems Russsel and Charles [1] were the first to study this lubrication effect by means of mathematical modeling. For the two-phase laminar concentric-annular flow in a horizontal pipe, they found a simple analytical solution, similar to the classical solution of the Navier–Stokes equations for the Hagen–Poiseuille flows. In their study, the interface is a closed circle concentric to the walls of the circular pipe. Evidently, due to the interaction of gravity and capillarity forces, there can exist more complex interface configurations, which were studied in a significant number of further works in which the following types of interfaces have been considered:

Type 1. Concentric circular interfaces (as in [1]).
Type 2. Eccentric circular interfaces. The interface is a closed circle eccentric to the walls of the circular pipe.
Type 3. Planar, side-by-side interfaces. The interface is a horizontal segment connecting the triple points on the left and right walls of the pipe.
Type 4. Circular, side-by-side interfaces. The interface is a circular arc connecting the triple points (see Fig. 1b).
Type 5. Fully eccentric circular interfaces. The interface is a closed circle, which touches the wall of the circular pipe.

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